Exact Solutions on Twisted Rings for the 3D Navier-Stokes Equations

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Abstract

The problem of describing the behavior of the solutions to the Navier-Stokes equations in three space dimensions has always been borderline. From one side, due to the viscosity term, smooth data seem to produce solutions with an everlasting regular behavior. On the other hand, the lack of a convincing theoretical analysis suggests the existence of possible counterexamples. In particular, one cannot exclude the blowing up of solutions in finite time even in presence of smooth data. Here we give examples of explicit solutions of the non-homogeneous equations. These are defined on a Hill's type vortex where the flow is rotating and swirling at the same time, inducing the flux to spiraling at a central node. Despite the appearance, the solution still remains very regular at the agglomeration point. The analysis may lead to a better understanding of the subtle problem of characterizing the solution space of the 3D Navier-Stokes equations. For instance, this result makes more narrow the path to the search of counterexamples built on the stretching and twisting of vortex tubes.

1   An instructive set of explicit solutions

The literature on fluid dynamics offers a wide range of exact solutions, suitable to almost any kind of applications (see for instance [13] for a review). Many examples are built starting from *vector spherical harmonic* functions, which are generally used as expansion basis for more advanced scenarios (see, e.g., [12]). Here we deal with some of these functions, which have not to be confused

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with their simpler version related to the *scalar* Laplace equation in spherical coordinates (see for instance [2], p.121). For this reason, this section is going to be a bit technical. We discuss a specific situation from the fluid dynamics viewpoint, in order to prepare the ground to generalizations that will allow in section 3 to construct families of solutions defined on evolving vortex rings.

First of all, we introduce the family of Bessel functions of the first kind:

$$J_\alpha''(r) \; + \; \frac{J_\alpha'(r)}{r} \; - \; \alpha^2 \frac{J_\alpha(r)}{r^2} \; = \; - J_\alpha(r) \qquad r \geq 0 \tag{1.1}$$

where, in our context, α is a real positive parameter. It is known that, for r tending to zero, the following asymptotic estimate holds:

$$J_\alpha(r) \; \approx \; \frac{1}{\Gamma(\alpha+1)} \left(\frac{r}{2}\right)^\alpha \tag{1.2}$$

where Γ is the Gamma function. Moreover, J_α turns out to be bounded, since it decays to zero as $\sqrt{2/\pi r}$, for $r \to +\infty$, with an oscillating behavior. For other properties on Bessel functions the reader is addressed to [14].

Successively, it is necessary to introduce the set of Legendre polynomials:

$$(1 - x^2)P_n''(x) \; - \; 2xP_n'(x) \; + \; n(n+1)P_n(x) \; = \; 0 \qquad x \in [-1, 1] \;\; n \geq 0 \tag{1.3}$$

By the substitution $x = \cos\theta$, the above equation takes the form:

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin^2\theta \; P_n'(\cos\theta) \right) \; = \; n(n+1)P_n(\cos\theta) \tag{1.4}$$

We are now ready to work in spherical coordinates:

$$(x, \; y, \; z) \; = \; (r\sin\theta\cos\phi, \; r\sin\theta\sin\phi, \; r\cos\theta) \tag{1.5}$$

with $0 \leq \phi < 2\pi$, $0 \leq \theta \leq \pi$ and $r \geq 0$.

Let us start by fixing $n \geq 1$ and defining the stationary vector field $\hat{\mathbf{w}} = (\hat{w}_1, \hat{w}_2, \hat{w}_3)$ whose components are:

$$\hat{w}_1 \; = \; n(n+1)\frac{J_{n+1/2}(r)}{r\sqrt{r}} \; P_n(\cos\theta)$$

$$\hat{w}_2 \; = \; -\frac{1}{\sqrt{r}} \left(J_{n+1/2}'(r) \; + \; \frac{J_{n+1/2}(r)}{2r} \right) \sin\theta \; P_n'(\cos\theta)$$

$$\hat{w}_3 \; = \; 0 \tag{1.6}$$

Note that there is no dependence on the variable ϕ. Moreover, there is cylindrical symmetry along the z-axis. Later, in section 4, we will generalize this setting.

Afterwards, one defines the time-dependent field $\hat{\mathbf{v}} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)$ given by:

$$\hat{\mathbf{v}} \; = \; e^{-\nu t}\hat{\mathbf{w}} \qquad t \geq 0 \tag{1.7}$$

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The constant $\nu > 0$ will play the role of viscosity parameter.

We start by discussing the regularity of $\hat{\mathbf{w}}$. Of course, the same arguments will apply to $\hat{\mathbf{v}}$, $\forall t \geq 0$. It should be clear that $\hat{\mathbf{w}} \in C^\infty(\mathbf{R}^3 - \{0\})$, $\forall n \geq 1$. Less evident is the situation at the origin ($r = 0$). For $n \geq 2$, thanks to (1.2) we deduce that $\hat{\mathbf{w}}$ tends to zero. Therefore, a continuous prolongation is obtained by setting $\hat{\mathbf{w}} = 0$ at the origin. The differentiability follows from an analysis similar to that of the more delicate case $n = 1$, to which we are now going to concentrate our attention.

Let us first note that $P_1(\cos\theta) = \cos\theta$ and $P_1'(\cos\theta) = 1$. Moreover, it is known that:

$$J_{3/2}(r) = \sqrt{\frac{2}{\pi}} \frac{\sin r - r\cos r}{r\sqrt{r}} \tag{1.8}$$

Thus, in a neighborhood of the origin one gets:

$$n(n+1)\frac{J_{3/2}(r)}{r\sqrt{r}} \approx \frac{2}{3}\sqrt{\frac{2}{\pi}}\left(1 - \frac{r^2}{10} + \frac{r^4}{280}\right) \tag{1.9}$$

The second component of $\hat{\mathbf{w}}$ follows a similar asymptotic behavior:

$$\frac{1}{\sqrt{r}}\left(J_{3/2}'(r) + \frac{J_{3/2}(r)}{2r}\right) \approx \frac{2}{3}\sqrt{\frac{2}{\pi}}\left(1 - \frac{r^2}{5} + \frac{3r^4}{280}\right) \tag{1.10}$$

The important fact is that the coefficients in (1.9) and (1.10) not depending on r are the same.

In order to understand what is actually happening for $r = 0$ it is better to argue in Cartesian coordinates. The local orthogonal triplet of the spherical reference framework, has the expression:

$$\mathbf{r} = (\sin\theta\cos\phi, \ \sin\theta\sin\phi, \ \cos\theta)$$

$$\boldsymbol{\theta} = (\cos\theta\cos\phi, \ \cos\theta\sin\phi, \ -\sin\theta)$$

$$\boldsymbol{\phi} = (-\sin\phi, \ \cos\phi, \ 0) \tag{1.11}$$

with

$$\sin\theta = \sqrt{x^2+y^2}/\sqrt{x^2+y^2+z^2} \qquad \cos\theta = z/\sqrt{x^2+y^2+z^2}$$

$$\sin\phi = y/\sqrt{x^2+y^2} \qquad \cos\phi = x/\sqrt{x^2+y^2} \tag{1.12}$$

Neglecting higher order infinitesimal terms, near the origin one has from (1.9) and (1.10):

$$\hat{\mathbf{w}} = \hat{w}_1\mathbf{r} + \hat{w}_2\boldsymbol{\theta} + \hat{w}_3\boldsymbol{\phi} \approx \frac{2}{3}\sqrt{\frac{2}{\pi}}\left(\frac{zx}{10}, \ \frac{zy}{10}, \ 1 - \frac{z^2}{10} - \frac{x^2+y^2}{5}\right) \tag{1.13}$$

By placing at the origin the vector $(0, 0, 2\sqrt{2}/3\sqrt{\pi})$ (written in Cartesian coordinates) one gets a continuous vector field. A more careful analysis shows that $\hat{\mathbf{w}}$ can be differentiated infinite times. A plot of $\hat{\mathbf{w}}$ near the origin of the plane $\phi = 0$ is shown in figure 1. Due to the decay at infinity of Bessel functions, the whole field is bounded in $\mathbf{R}^3$.
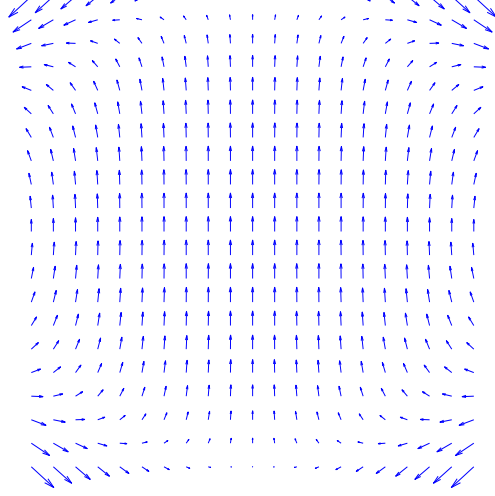
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Figure 1: Qualitative displacement of the field $\hat{\mathbf{w}}$ near the origin for $n = 1$. The plot shows the projection on the plane $(x, 0, z)$ obtained from (1.5) by setting $\phi = 0$.

The vector fields previously introduced satisfy a series of striking results that we are going to recall. The first important relation is that, independently of n, $\hat{\mathbf{v}}$ has zero divergence:

$$\text{div}\hat{\mathbf{v}} = 0 \quad \text{in } \mathbf{R}^3 \quad \forall\, t \geq 0 \qquad (1.14)$$

The proof is obtained by a direct check. Indeed, for $\alpha = n + 1/2$, thanks to (1.4) one recovers $\forall t \geq 0$ and $r > 0$:

$$\text{div}\hat{\mathbf{v}} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \hat{v}_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ \hat{v}_2)$$

$$= e^{-\nu t}\left[\frac{n(n+1)}{r^2}\frac{\partial}{\partial r}(\sqrt{r}J_\alpha)\ P_n(\cos\theta)\right.$$

$$\left. - \frac{1}{r\sqrt{r}\sin\theta}\left(J_\alpha' + \frac{J_\alpha}{2r}\right)\frac{\partial}{\partial\theta}\left(\sin^2\theta\ P_n'(\cos\theta)\right)\right] = 0 \qquad (1.15)$$

Equation (1.14) is also true at the origin. To check this, one can use either (1.13) or symmetry arguments (from the physical viewpoint in any neighborhood of the origin the inbound flow equates that directed outbound).

Another important fact is that $\hat{\mathbf{v}}$ satisfies the parabolic equation:

$$\frac{\partial\hat{\mathbf{v}}}{\partial t} - \nu\Delta\hat{\mathbf{v}} = 0 \qquad \forall\, t \geq 0 \qquad (1.16)$$

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The statement follows from writing the Laplace operator in spherical coordinates. Considering that there is no dependence on the variable ϕ, the expression of the operator looks rather simplified:

$$(\Delta\hat{\mathbf{v}})_1 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\hat{v}_1}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\hat{v}_1}{\partial\theta}\right)\right.$$

$$\left. - 2\left(\hat{v}_1 + \frac{\partial\hat{v}_2}{\partial\theta} + \hat{v}_2\frac{\cos\theta}{\sin\theta}\right)\right] = \frac{1}{\nu}\frac{\partial\hat{v}_1}{\partial t} \qquad (1.17)$$

$$(\Delta\hat{\mathbf{v}})_2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\hat{v}_2}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\hat{v}_2}{\partial\theta}\right)\right.$$

$$\left. + 2\frac{\partial\hat{v}_1}{\partial\theta} - \frac{\hat{v}_2}{\sin^2\theta}\right] = \frac{1}{\nu}\frac{\partial\hat{v}_2}{\partial t} \qquad (1.18)$$

$$(\Delta\hat{\mathbf{v}})_3 = \frac{1}{\nu}\frac{\partial\hat{v}_3}{\partial t} = 0 \qquad (1.19)$$

The check of the above formulas, by direct substitution of the expressions of $\hat{v}_1$ and $\hat{v}_2$, requires however a generous amount of patience. We do not report such details here. We just recall that the equations (1.1) and (1.3) must be used with profusion. A simpler verification is obtained by recovering a potential vector $\mathbf{A}$ such that $\hat{\mathbf{w}} = \text{rot}\mathbf{A}$ (see section 3) and showing that $\mathbf{A} = \text{rot}\hat{\mathbf{w}}$. This implies that $-\Delta\hat{\mathbf{w}} = \text{rot}(\text{rot}\hat{\mathbf{w}}) = \hat{\mathbf{w}}$ (recall that $\hat{\mathbf{w}}$ has zero divergence), and this last relation implies (1.16).

Similar solutions can be found in the 2D framework. For instance, working in polar coordinates (r, θ), we define for $n \geq 1$:

$$\hat{\mathbf{w}} = (\hat{w}_1, \hat{w}_2) = \left(n^2\frac{J_n(r)}{r}T_n(\cos\theta), \ -J_n'(r)\sin\theta\ T_n'(\cos\theta)\right) \qquad (1.20)$$

where T_n is the n-th Chebyshev polynomial. Again, one has $\text{div}\hat{\mathbf{w}} = 0$ and $-\Delta\hat{\mathbf{w}} = \hat{\mathbf{w}}$.

After having introduced the velocity field $\hat{\mathbf{v}}$, we need to explain its role in the framework of the Navier-Stokes equations. This will be the subject of the next section. The origin must be regarded as the bottleneck of a generous amount of fluid entering for $z < 0$ and exiting for $z > 0$. Nevertheless, this is not enough to generate singularities in the field or its derivatives. Such an observation continues to be true even by introducing a swirl in the ϕ direction, as we shall see in section 4.

2 Connections with the Navier-Stokes equations

A further property of the field $\hat{\mathbf{w}}$ is that its rotational has a very simple expression:

$$\text{rot}\hat{\mathbf{w}} = \left(0, \ 0, \ \frac{1}{\sqrt{r}}J_{n+1/2}(r)\sin\theta\ P_n'(\cos\theta)\right) \qquad (2.21)$$

where the third component has been computed via the formula:

$$(\mathrm{rot}\hat{\mathbf{w}})_3 \;=\; \frac{1}{r}\left[\frac{\partial}{\partial r}(r\hat{w}_2) \;-\; \frac{\partial \hat{w}_1}{\partial \theta}\right] \tag{2.22}$$

Due the asymptotic expression (1.2), the rotational goes smoothly to zero at the origin as r^n. Considering that $\hat{\mathbf{w}}$ is bounded, the field $\hat{\mathbf{w}} \times \mathrm{rot}\hat{\mathbf{w}}$ also goes to zero at the origin.

We now set:

$$\hat{\mathbf{f}} \;=\; -\,\hat{\mathbf{v}} \times \mathrm{rot}\hat{\mathbf{v}} \;=\; -\; e^{-2\nu t}\,\hat{\mathbf{w}} \times \mathrm{rot}\hat{\mathbf{w}}$$

$$\hat{p} = -\tfrac{1}{2}|\hat{\mathbf{v}}|^2 \;=\; -\;\tfrac{1}{2}e^{-2\nu t}\,|\hat{\mathbf{w}}|^2 \tag{2.23}$$

By applying well-known notions of differential calculus, one may write:

$$(\hat{\mathbf{v}} \cdot \nabla)\hat{\mathbf{v}} \;=\; \tfrac{1}{2}\nabla|\hat{\mathbf{v}}|^2 \;-\; \hat{\mathbf{v}} \times \mathrm{rot}\hat{\mathbf{v}} \;=\; -\nabla\hat{p} \;+\; \hat{\mathbf{f}} \tag{2.24}$$

Therefore, using (1.16), we finally have:

$$\frac{\partial \hat{\mathbf{v}}}{\partial t} \;-\; \nu\Delta\hat{\mathbf{v}} \;+\; (\hat{\mathbf{v}} \cdot \nabla)\hat{\mathbf{v}} \;=\; -\nabla\hat{p} \;+\; \hat{\mathbf{f}} \qquad \forall\, t \geq 0 \tag{2.25}$$

Recalling the divergence-free condition (1.2), we may argue that the couple $\hat{\mathbf{v}}$, $\hat{p}$ satisfies the 3D Navier-Stokes problem with the forcing term $\hat{\mathbf{f}}$ and the initial condition $\hat{\mathbf{v}} = \hat{\mathbf{w}}$ at time $t = 0$. Note that our pressure is negative, but this is not a problem, since a scalar potential is defined up to an additive constant and $|\hat{\mathbf{v}}|$ is bounded in the whole $\mathbf{R}^3$ space.

The choice of $\hat{p}$ looks physically correct. The quantity $\frac{1}{2}\rho|\hat{\mathbf{v}}|^2$, where ρ is the mass density, is related to kinetic energy. In the incompressible case, which is the one we are studying here, ρ is constant in $\mathbf{R}^3$ (assume $\rho = 1$). The setting is in agreement with the Bernoulli principle, stating that the quantity $\frac{1}{2}\rho|\hat{\mathbf{v}}|^2 + \hat{p}$ is preserved along the stream-lines. Thus, the balance between $-\nabla\hat{p}$ and the gradient part of $(\hat{\mathbf{v}} \cdot \nabla)\hat{\mathbf{v}}$ only involves energy properties belonging to the fluid, while the extra-force $\hat{\mathbf{f}}$ (not of conservative type in our case) represents external constraints.

Including potential functions of conservative fields, such as gravity, in the pressure term is an allowed procedure (in this fashion, when considering the case of gravitational potentials, pressure directly turns out to decay with altitude). Effective forcing terms can be built with fields having the rotational different from zero. Note for instance that, if (2.23) was replaced by the setting $\hat{\mathbf{f}} = 0$ and $(\hat{\mathbf{v}} \cdot \nabla)\hat{\mathbf{v}} = -\nabla\hat{p}$, we would get an incompatible situation since the advective term is not the gradient of any potential.

Our solution is defined in $\mathbf{R}^3$. Of course, a fictitious boundary-value problem on an open domain Ω may be obtained by assigning on $\partial\Omega$ the restriction of the field $\hat{\mathbf{v}}$, $\forall t > 0$. When Ω is a sphere of radius R centered at the origin, there are two interesting situations to be mentioned. If R is a zero of J_α ($\alpha = n + 1/2$),

we have that the radial component on $\partial\Omega$ is identically zero, so the boundary field is tangential. The boundary datum is purely radial if instead R is a zero of $J'_\alpha + J_\alpha/2r$.

Assuming that R is the first zero of $J_{n+1/2}$ (the most interesting case is $n = 1$, where R is approximately equal to 4.49), we can interpret the solution as a stable vortex ring Ω. Tangential velocity is imposed on the boundary and the term $\hat{\mathbf{f}}$ plays the role of centripetal force keeping the fluid in stationary circular motion. In truth the domain Ω is a degenerate toroid where the central hole is reduced to a segment of the vertical axis z, i.e., we are simulating a Hill's type vortex (see, e.g., [1], p.175).

The same construction can be considered in the 2D case (see (1.20)). The corresponding $\hat{\mathbf{f}}$ is given by $J_n(r)\sin\theta\, T'_n(\cos\theta)(-\hat{w}_2, \hat{w}_1)$, which is tending to zero at the origin (see (1.2)) for any $n \geq 1$.

Let us also note that in order to get a time decay at infinity for the solution proportional to $e^{-\nu t}$, as the dissipative term prescribes, our forcing field $\hat{\mathbf{f}}$ must go to zero at least as $e^{-2\nu t}$, i.e. quadratically faster. This observation indicates that the time smoothness properties of the right-hand side should enter more formally in the a priori estimates, if one looks for regularity theorems.

In the theoretical analysis of the Navier-Stokes equations, the gradient terms are usually eliminated by projection into the space of functions with null divergence. An alternative approach is to differentiate the momentum equation. By taking the rotational on both sides, one gets for some right-hand side $\mathbf{f}$:

$$\frac{\partial\boldsymbol{\omega}}{\partial t} \;-\; \nu\Delta\boldsymbol{\omega} \;+\; (\mathbf{v}\cdot\nabla)\boldsymbol{\omega} \;-\; (\boldsymbol{\omega}\cdot\nabla)\mathbf{v} \;=\; \mathrm{rot}\mathbf{f} \qquad (2.26)$$

where the vorticity $\boldsymbol{\omega}$ automatically satisfies $\mathrm{div}\boldsymbol{\omega} = 0$. In this way we again eliminate the gradient term $\nabla(\frac{1}{2}|\mathbf{v}|^2 + p)$. Note that $\boldsymbol{\omega}$ does not satisfy homogeneous boundary conditions, making difficult the theoretical analysis of (2.26).

From the explicit expression of $\hat{\mathbf{w}}$, a certain number of collateral problems can be examined. For instance, one can try to find $\mathbf{v}$ and p such that:

$$\frac{\partial\mathbf{v}}{\partial t} \;-\; \nu\Delta\mathbf{v} \;+\; (\mathbf{v}\cdot\nabla)\mathbf{v} \;=\; -\nabla p \;+\; \mathbf{f} \qquad \forall t > 0$$

$$\mathrm{div}\mathbf{v} \;=\; 0 \quad \forall t > 0 \qquad \text{and} \;\; \mathbf{v} = \mathbf{v}_0 = 0 \;\; \text{for} \;\; t = 0 \qquad (2.27)$$

Here $\mathbf{f} = -\gamma\hat{\mathbf{w}} \times \mathrm{rot}\hat{\mathbf{w}}$, where $\hat{\mathbf{w}}$ is defined in (1.6) and γ is a given positive function of time with $\gamma(0) = 0$. We suppose that γ grows for a while, then it decays fast to zero. Both $\mathbf{f}$ and $\mathbf{v}_0$ are defined in $\mathbf{R}^3$. The solution $\mathbf{v}$ is expected to develop the same vortices as the field $\hat{\mathbf{v}}$ introduced before. This should happen as soon as $t > 0$. Things may however depend on the rate of growth of γ. We guess (but this is just a conjecture to be verified with numerical simulations) that, if γ follows a slow growth at the beginning, viscosity might dominate smoothing out the solution. Alternatively, if γ has a rapid increase, the rotatory behavior may catch up and persist.

A variant of the example above is reformulated by replacing $\mathbf{f}$ in (2.27) by a uniformly approximating sequence $\mathbf{f}_\tau$ of functions. We can let the parameter τ depend on t, in such a way that the limit $\mathbf{f} = -\hat{\mathbf{w}} \times \mathrm{rot}\hat{\mathbf{w}}$ is reached in a finite time T. It is possible to define $\mathbf{f}_\tau$ in the time interval $]0, T[$ in order to get solutions representing classical vortex rings (i.e., closed vortex tubes shaped like a doughnut) tending to a Hill's type vortex as $t \to T$. We study in the next section a general recipe to generate this kind of sequences.

3   Solutions on evolving rings

Let us now introduce a general approach to get families of solutions defined on rings. We first observe that a vector potential $\mathbf{A}$ may be assigned to the field $\hat{\mathbf{w}}$ in (1.6). This is such that $\hat{\mathbf{w}} = \mathrm{rot}\mathbf{A}$ and explicitly given by $\mathbf{A} = (0, 0, A_3)$, where:

$$A_3 = \frac{1}{\sqrt{r}} J_{n+1/2}(r) \sin\theta \; P_n'(\cos\theta) \qquad (3.28)$$

Information on vector potentials can be for instance retrieved in [7], p.53, where connections with magnetic fields are also mentioned. It turns out that $\mathbf{A}$ is zero along the z-axis.

Comparing with (2.21), we deduce that $\mathbf{A} = \mathrm{rot}\hat{\mathbf{w}}$. As a consequence, by observing that $\mathrm{rot}(\mathrm{rot}\mathbf{A}) = -\Delta\mathbf{A} + \nabla(\mathrm{div}\mathbf{A}) = -\Delta\mathbf{A}$, the following relation holds:

$$-\;\Delta\mathbf{A} \;=\; \mathbf{A} \qquad (3.29)$$

Therefore $\mathbf{A}$ is an eigenfunction of the vector Laplacian, corresponding to the eigenvalue $\lambda = 1$. More precisely, the sphere Ω centered at the origin of radius R (the first zero of $J_{n+1/2}$), deprived of the vertical axis, is such that the smallest eigenvalue of the operator $-\Delta$ is equal to 1. The homogeneous boundary conditions associated to such an operator are of Dirichlet type, including in the boundary also the vertical segment at the intersection of the sphere and the z-axis. They can be however substituted by tangential type conditions:

$$\mathrm{rot}\mathbf{A} \cdot \mathbf{n} \;=\; 0 \qquad (3.30)$$

with $\mathbf{n}$ denoting the outward normal field on $\partial\Omega$, that is what we got for the field $\hat{\mathbf{w}}$.

In order to examine other similar situations, we work with cylindrical coordinates (r, z, ϕ). We look for vector potentials of the form $\mathbf{A} = (0, 0, A_3)$, where A_3 does not depend on ϕ. In addition we would like to impose (3.29) and (3.30). We are then requiring that $\lambda = 1$ must be the first eigenvalue, so restricting a bit the choice of the domain Ω. In terms of the only unknown A_3, the vector differential problem becomes:

$$-\left(\frac{\partial^2 A_3}{\partial r^2} \;+\; \frac{1}{r}\frac{\partial A_3}{\partial r} \;+\; \frac{\partial^2 A_3}{\partial z^2} \;-\; \frac{A_3}{r^2} \right) \;=\; A_3 \qquad (3.31)$$

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One can easily check that the two other components of $\Delta\mathbf{A}$ are zero. The scalar equation (3.31) is either subject to homogeneous Dirichlet boundary conditions or to the boundary constraint:

$$\frac{\partial A_3}{\partial z}n_1 = \left(\frac{\partial A_3}{\partial r} + \frac{A_3}{r}\right)n_2 \tag{3.32}$$

The above equations can be simplified further by setting $B = rA_3$. By this substitution our problem takes the form:

$$-\left(\frac{\partial^2 B}{\partial r^2} - \frac{1}{r}\frac{\partial B}{\partial r} + \frac{\partial^2 B}{\partial z^2}\right) = B \quad \text{in } \Xi \qquad B = 0 \ \text{ on } \partial\Xi \tag{3.33}$$

where Ξ is the section of the toroid region Ω. Note that B is determined up to multiplicative constants (related to the intensity of the final velocity field). On the left-hand side of (3.33) we have a positive definite operator, as a result of little manipulation after multiplying by B/r and integrating on Ξ.

Since there is no dependence on ϕ, we can simulate flows on vortex rings having a symmetry axis coincident with the z-axis. An analysis of this kind has been considered in [5] from a different perspective, where inviscid fluids are coupled with electromagnetism. Note that the example introduced in section 1 for $n = 1$ is related to famous Hertz solution, aimed to describe the electromagnetic field emanated by an infinitesimal dipole. Extensions, examined through a series of numerical tests, have been successively investigated in [3]. The results and the techniques there presented may be suitably adapted to the study of vortex formation and stability.

For example, let Ξ be the circle of radius R, situated in the plane (r, z) and centered at a point of the semi-axis $r > 0$ very far away from the origin. By neglecting the terms in (3.31) containing r at the denominator, the function A_3 solves the 2D Laplace eigenvalue equation. The minimum eigenvalue in Ξ is approximately equal to $(\xi/R)^2$ (where $\xi \approx 2.4$ is the first zero of J_0), so that, by taking $R \approx 2.4$, we get a reasonably good solution of problem (3.29). Indeed, the 3D ring Ω with circular section Ξ and vertical axis z is a domain where the equation $-\Delta\mathbf{A} = \mathbf{A}$ can be almost perfectly solved in terms of orthogonal basis functions. There are infinite other rings, more or less of the same shape and size as Ω, where (3.29) holds once one knows the eigenfunctions of (3.33). When the section Ξ approaches the origin, we can flatten the part of it facing the z-axis, while preserving the minimum eigenvalue $\lambda = 1$ (see figure 2). The result is the transformation of standard ring into a Hill's type one.

The flow in Ω may be defined by setting $\mathbf{v} = e^{-\nu t}\text{rot}\mathbf{A}$. Thus, as done in section 2, the full set of Navier-Stokes equations (including the div$\mathbf{v} = 0$ condition) is satisfied by writing:

$$\frac{\partial\mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \tfrac{1}{2}\nabla|\mathbf{v}|^2 - \mathbf{v}\times\text{rot}\mathbf{v} = -\nabla p + \mathbf{f} \tag{3.34}$$

where the gradient term on the right-hand side has been assimilated to a gradient of pressure, while the rest is a forcing source. This last centripetal acceleration

is the one needed to maintain the flow constrained in the ring Ω. The energy of the flow is exponentially fading in time due to viscosity. In the above mentioned procedure it is not clear how to prolong the field $\mathbf{v}$ outside Ω. Realistically, the external fluid is dragged by viscous effects. The fluid across the boundary $\partial\Omega$ is continuous but may lose regularity there. This is the same difficulty one may find in the study of typhoons, at the interface between the central core and the peripheral dragged flow.

The request that the eigenvalue λ remains equal to a given constant is quite important in the framework of electromagnetic applications (where $\lambda = c^2$ and c is the speed of light), but it is not crucial in fluid dynamics. We just have to pay attention when passing to the parabolic equation $(\partial\mathbf{v}/\partial t) = \nu\Delta\mathbf{v}$. For any fixed t, let $\mathbf{w}$ be the stationary solution satisfying: $-\Delta\mathbf{w} = \lambda(t)\mathbf{w}$. As previously done, one sets $\mathbf{v} = \gamma(t)\mathbf{w}$. The function γ has to be then determined according to the relation:

$$\frac{d\gamma}{dt} = -\nu\lambda\gamma \quad \text{for} \ \ t \in]0, T[\tag{3.35}$$

yielding the exponential $\gamma(t) = e^{-\nu t}$ in the particular case $\lambda = 1$.
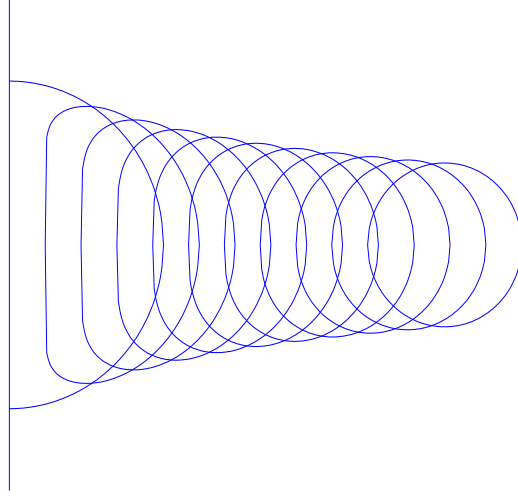

Figure 2: Sections of evolving tori. Starting from a classical ring having circular section (right), one can deform the shape and obtain through a sequence of intermediate rings a spherical doughnut (left) where the hole is reduced to a vertical segment.

A classical way of looking for troubled solutions of the Navier-Stokes equations is to let evolve regular vortex tubes in order to create a singularity at some

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point. Considerations about this hypothesis are reported for instance in [6], [8], [9], [10] and [11], section 5.4. Some theoretical implications are examined in [4].

Therefore, from our viewpoint, one can imagine a smooth ring driven by an internal force $\mathbf{f}_\tau$ changing with time. In a finite time, the process ends up with the flow constricted into a Hill's vortex, where the set of equations are actually solved on a spherical domain. Note that, due to incompressibility, the volume of these rings must remain constant. This is also true because we are imposing tangential boundary conditions (see (3.30)), therefore the fluid cannot escape.

In the plane (r, z), a possible transition from a classical vortex ring displaying circular section into a spherical vortex is shown in figure 2. The evolution of the boundary may be described for $t \in [0, 1]$ by the formula:

$$\left(\sqrt{(r - s(t))^2 + z^2} \right)^{2-t} = (r - s(t))^{1-t} \tag{3.36}$$

where $s > 0$ is a time-dependent decreasing function satisfying s(1)=0. For $t = 0$, we have the circumference of radius $\frac{1}{2}$ centered at $(\frac{1}{2} + s(0), 0)$. After a sudden change of topology, for $t = 1$ we obtain the semi-circumference of radius 1 centered at the origin. This construction is only given to qualitatively illustrate the situation. We do not know for example if the family of rings built by rotating the domains of figure 2 around the z-axis preserves the eigenvalue $\lambda = 1$ as prescribed by (3.29).

It seems however that no irregularity turns out to be present at the end of this process, since the solution presented in section 2 is smooth up to the origin. This does not prove that any attempt to collapse a ring hole into a point (or a segment), according to the equations, is going to fail. Nevertheless, the percentage of success in the search of anomalous solutions of this type becomes very low.

4   A Hill's type vortex ring with twist

We finish by retouching the solution defined in section 2. We now add a component along the ϕ direction. Thus, we go back to spherical coordinates (r, θ, ϕ) and recall the field $\hat{\mathbf{w}}$ defined in (1.6). For simplicity we only discuss the case $n = 1$. We now define:

$$\hat{\mathbf{w}} = \left(\hat{w}_1, \ \hat{w}_2, \ \sigma \frac{J_{3/2}(r)}{\sqrt{r}} \sin\theta \right) \tag{4.37}$$

where σ is a given number. Practically, the stationary flow, while behaving as in figure 1, also moves around the vertical axis. The third component in (4.37) is zero for $r = 0$ by virtue of (1.2). By computing the rotational, we now get:

$$\mathbf{A} = \text{curl}\hat{\mathbf{w}} = \left(\frac{2\sigma J_{3/2}(r)}{r\sqrt{r}} \cos\theta, -\frac{\sigma}{\sqrt{r}} \left(J'_{3/2}(r) + \frac{J_{3/2}(r)}{2r} \right) \sin\theta, \ A_3 \right) \tag{4.38}$$

where A_3 is given in (3.28). It turns out that $\hat{\mathbf{w}} = \text{curl}\mathbf{A}$, so we are dealing with the same kind of solutions previously studied. Such solutions are defined everywhere. However, by denoting with R the first zero of $J_{3/2}$ (which is approximately equal to 4.49), the solution can be restricted to the sphere centered at the origin of radius R.

Though $\hat{\mathbf{w}}$ still does not depend on ϕ, the modified flow field twists in the direction of the vector $\boldsymbol{\phi}$ (see figure 3). Note that the boundary conditions to be assigned on the surface of the sphere of radius R only contain the component along $\boldsymbol{\theta}$. As figure 3 indicates, the field has only a vertical (along the z-axis) component at the origin. In the immediate surroundings, a component along $\boldsymbol{\phi}$ develops, which can be very strong in magnitude since its strength depends on σ, which is arbitrary. Therefore, sharp layers are observed near the origin, although the solution continues to be smooth. Together with the origin, other critical regions of sharp layer formation are at the poles of the sphere.

The trajectory of a particle is practically untwisted near the external boundary. In this situation the orbits are mildly precessing. Nevertheless, as the particle gets to the pole, is pushed towards the center and starts spiraling until it reemerges near the opposite pole. For particles placed inside the sphere far from the axis and the boundary, the orbits tend to follow a helicoid with circular section, turning around the z-axis. There might be no periodicity in this complex motion.
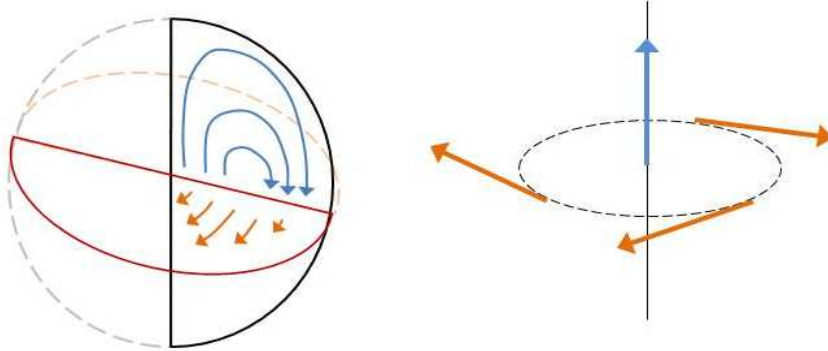

Figure 3: The field of the new spherical vortex combines the component describing the rotation within each section with a perpendicular one following closed circular patterns around the vertical axis. The horizontal velocities are different depending on the distance from the origin. In the neighborhood of the origin (picture on the right), the trajectories pass with continuity from a pure vertical motion to spirals. The variation can be particularly significant, especially when σ is large.

When we compute the product $\hat{\mathbf{w}} \times \mathrm{rot}\hat{\mathbf{w}}$, it should be evident that its first two components tend to zero for $r \to 0$. The third one takes the form:

$$(\hat{\mathbf{w}} \times \mathrm{rot}\hat{\mathbf{w}})_3 \;=\; w_1 A_2 \;-\; w_2 A_1 \tag{4.39}$$

A careful analysis shows that this also tends to zero for $r \to 0$. We can argue locally in the neighborhood of the origin by working in Cartesian coordinates. In this way, excluding third-order terms, the expression given in (1.13) has to be modified as follows:

$$\hat{\mathbf{w}} \;\approx\; \frac{2}{3}\sqrt{\frac{2}{\pi}} \left(\frac{zx}{10} - \sigma y, \; \frac{zy}{10} + \sigma x, \; 1 - \frac{z^2}{10} - \frac{x^2+y^2}{5} \right) \tag{4.40}$$

Considering that $\mathrm{rot}\hat{\mathbf{w}} \approx (2\sqrt{2}/3\sqrt{\pi})(-y/2, x/2, 2\sigma)$, one ends up with the approximation:

$$-\nu\Delta\hat{\mathbf{w}} - \hat{\mathbf{w}} \times \mathrm{rot}\hat{\mathbf{w}} \;\approx\; \frac{8}{9\pi} \left(\frac{(4\sigma^2 - 1)x}{2} + \frac{\sigma yz}{5} + \frac{xz^2}{20} + \frac{x}{10}(x^2+y^2), \right.$$

$$\left. \frac{(4\sigma^2 - 1)y}{2} - \frac{\sigma xz}{5} + \frac{yz^2}{20} + \frac{y}{10}(x^2+y^2), \; \frac{z}{20}(x^2+y^2) + \nu\frac{3}{2}\sqrt{\frac{\pi}{2}} \right) \tag{4.41}$$

The special choice $\sigma = 1/2$, that allows the above terms to tend to zero faster at the origin, worth to be recorded. For this value of σ, the divergence of $\hat{\mathbf{w}} \times \mathrm{rot}\hat{\mathbf{w}}$ is zero at the origin.

An interesting collateral problem to be studied is when the initial solution of the homogeneous ($\mathbf{f} = 0$) Navier-Stokes equations it taken to be equal to $\hat{\mathbf{w}}$. In addition, the flow is constantly supplied at the boundary of the sphere of radius R with the stationary values of $\hat{\mathbf{w}}$. In absence of the forcing term it is reasonable to expect a decay to zero at the origin due to viscosity. This should be true independently of σ, but strange surprises could also happen. It would be wise to run some numerical tests to verify the behavior.

Of course, one can add a twisting also to the annular flows introduced in section 3. In cylindrical coordinates it is enough to search for solutions of the form $\hat{\mathbf{w}} = (0, 0, \hat{w}_3)$, where $\hat{w}_3$ does not depend on ϕ. If we require that simultaneously $\mathbf{A} = \mathrm{rot}\hat{\mathbf{w}}$ and $\hat{\mathbf{w}} = \mathrm{rot}\mathbf{A}$, we arrive at the equation:

$$-\left(\frac{\partial^2 \hat{w}_3}{\partial r^2} \;+\; \frac{1}{r}\frac{\partial \hat{w}_3}{\partial r} \;-\; \frac{\hat{w}_3}{r^2} \;+\; \frac{\partial^2 \hat{w}_3}{\partial z^2} \right) \;=\; \hat{w}_3 \tag{4.42}$$

to be solved in the section Ξ. The solution is determined up to a constant σ. By imposing homogeneous Dirichlet boundary conditions we get that the flow is swirling only at the interior of the ring. For a ring with small circular section with respect to the major diameter, the solution of the full Navier-Stokes equations leads to a component along the ϕ direction similar to that of a Poiseuille type flow. As before, the section Ξ may be deformed depending on time as suggested by the plots of figure 2, so ending up with a configuration as the one depicted in figure 3.

In conclusion, we proposed a set of explicit solutions of the non-homogeneous Navier-Stokes equations expressed in spherical coordinates with the help of Bessel functions. These are defined in the whole 3D space, but they can be easily restated in a spherical domain subject to suitable tangential conditions. The flow rotates forming a Hill's vortex remaining smooth up to the inner point. A twist may be also introduced, but this does not alter the regularity of the solution. This example suggests that the search of blowing up solutions of the 3D Navier-Stokes equations is not going to be easy, since most of the counterexamples are built on the idea of crashing vortex tubes in order to detect possible degenerations at some meeting point. Together with the mentioned examples, we also proposed a way to explicitly construct vortex rings, that can be helpful in various other circumstances, both for theoretical and computational aspects.

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